A SIMPLE PROOF OF THE ASYMPTOTIC NORMALITY OF UNIVARIATE L-ESTIMATES AND APPLICATION

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ABSTRACT

This paper presents a simple proof of the asymptotic normality of L-estimates in the iid case. It then proceeds to discuss the generalization to the non-iid case. Certain restrictions have been placed on the weight function to avoid the requirement that the second moment of the underlying distribution be finite. An application is given in the regression setting. The approach taken uses only basic concepts in analysis, specifically, convergence theorems.

KEYWORDS: L-estimates, iid, non-iid, trimmed mean, robustness, asymptotic normality.

1. Introduction

Interest in L-estimates in the statistical literature lies mainly on the desirable robustness properties that this class of estimators exhibit. L estimates are defined as estimates of the form:

$$L_n = \sum_{i=1}^{n} c_i \, x(i) \qquad (1)$$

where c_i are weights, $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ are the ordered values of the observations $x_1, \ldots, x_n$ from a common distribution F(.).

Moore (1968) showed that the asymptotic distribution of the properly normalized estimates in the form of (1) is normal. However, Stigler (1974) showed that Moore's (1968) proof was faulty.

In this paper, we will attempt to establish the asymptotic normality of L_n using basically the idea of Moore (1968) and correcting the flaws in his arguments. Our proof is different from other proofs found in the literature utilizing only elementary concepts from calculus. A simple application of the estimate L_n is given in the regression setting.

2. L-Estimates

We consider two cases in order, as the sample values are i.i.d. or non-i.i.d.

2.1. Common Distribution

Let $x_1, x_2, \ldots, x_n$ be a sample from a distribution F and let $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ denote the ordered values in the sample. A linear function of the ordered values, given by:

$$L_n = \sum_{i=1}^{n} c_i x(i) \qquad (2)$$

where $c_1, c_2, \ldots, c_n$ are constants is called an L-estimate. It is convenient to write (2) as:

$$Ln = \frac{1}{n} \sum_{i}^{n} J\left(\frac{i}{n+1}\right) x(i)$$

where $J(u)$, $0 \leq u \leq 1$ represents a weight function. Some authors use

$$c_i = n \int_{\frac{i-1}{n}}^{1/n} J(u)du \qquad (3)$$

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in place of $J(\frac{1}{n+1})$ assuming that J is integrable. The corresponding L-estimate given by:

$$L_n'' = \frac{1}{n}\sum_{i=1}^{n} c_i x_{(i)}$$

$$= \int_0^1 \hat{F}_n^{-1}(t)J(t)dt$$

is thus formulated as a statistical function, where $\hat{F}_n(\cdot)$ denotes the sample cdf. It may be noted that

$$\max\left|J(\frac{i}{n+1}) - n\int_{\frac{i-1}{n}}^{\frac{1}{n}} J(t)dt\right| =$$

$$0(n^{-\alpha}),\ 1 \le i \le n$$

if J is Lipschitz of order α.

We now make the following assumptions. Let J be bounded and continuous a.e F^{-1} on $[0,1]$. Let

$$\int_{-\infty}^{\infty}|F(x)\ (1) - F(x))|^{\frac{1}{2}}dx < \infty \cdot \qquad (4)$$

If F has regularly varying tails then (4) is equivalent to the condition $E|x_j|^2 < \infty$. In any case, (4) implies that $E\ x_j|^2 < \infty$. (see Feller(1965), section V.6, Lemma 1). Let

$$\mu^2 = \int_{-\infty}^{\infty} x\ J(F)(x))dF(x),$$

and

$$\sigma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))\cdot$$

$$(\min\ [F(x),(F(y) - F(x))]\ F(y))dxdy$$

<u>Theorem 1</u>. Let J be bounded and continuous a.e. F^{-1} on $[0, 1]$. If (4) is satisfied and $\sigma^2 > 0$, then

$$\sqrt{n}(L_n'' - \mu) \xrightarrow{L} N(0, \sigma^2) \text{ as } n \to \infty \cdot$$

Proof: Let

$$\int_0^1 J(t)dt = c, \text{ and}$$

$$\Psi(u) = \int_u^1 J(t)dt - c(1 - \mu).$$

Note that, $\Psi(0) = \Psi(1) = 0$ and $\Psi'(u) = c - J(u)$. We have

$$u = \int_{-\infty}^{\infty} \Psi((F(x)dx + c\int_{-\infty}^{\infty} x\ dF(x), \text{ and}$$

$$L_n'' = \sum_1^n(\Psi(\frac{i-1}{n})) - \Psi(\frac{1}{n}))x_{(i)} + \frac{c}{n}\sum_1^n x_{(i)}$$

$$= \sum_1^{n-1}\Psi(\frac{i}{n})\ (x_{(i+1)} - x_{(i)}) + \frac{c}{n}\sum_1^n x_{(i)}$$

$$= \int_{-\infty}^{\infty}\Psi(\hat{F}_n(x))dx + c\int_{-\infty}^{\infty} x\ d\hat{F}_n(x).$$

Therefore,

$$\sqrt{n}\int_{-\infty}^{\infty}[\Psi|(\hat{F}_n(x)) - F(x)) - c(\hat{F}_n(x)) - F(x)]dx \quad (5)$$

By the Mean-Value Theorem,

$$\Psi(\hat{F}_n(x)) - \Psi(F(x)) =$$

$$\hat{F}_n(x), - F(x))(\Psi'(F(X)) + \lambda(\hat{F}_n(x) - F(x))$$

where $\lambda(x) \to 0$ as $x \to 0$.

Hence,

$$\sqrt{n}(L_n'' - \mu) = \sqrt{n}\int_{-\infty}^{\infty}([\hat{F}_n(x) - F(x)]\cdot$$

$$J(Fx)) - \lambda(\hat{F}_n(x) - F(x))dx \qquad (6)$$

Now,

$$E(\sqrt{n}\int_{-\infty}^{\infty}|F_n(x) - F(x)|dx \le$$

$$\sqrt{n}\int_{-\infty}^{\infty}(\ E(\hat{F}_n(x)) - F(x))^2\ )^{\frac{1}{2}}dx$$

$$\le \int_{-\infty}^{\infty}(F(x)\ (1-F(x)))^{\frac{1}{2}}dx < \infty \text{ by (4).}$$

Hence,

$$\sqrt{n}\int_{-\infty}^{\infty}|\hat{F}_n(x) - F(x)\ \ dx = 0_p(1)$$

Moreover, by the Glivenko-Cantelli Theorem, as $n \to \infty$

$$\sup_{-\infty < x < \infty}|\hat{F}_n(x) - F(x)| \xrightarrow{wpl} 0.$$

Therefore,

$$\sqrt{n}\int_{-\infty}^{\infty}(\hat{F}_n(x) - F(x))\lambda(\hat{F}_n(x) - F(x))dx \xrightarrow{P} 0 \qquad (7)$$

From (6) and (7) it follows that $\sqrt{n}(L_n'' - \mu)$

is asymptotically distributed as:

$$Z = -\sqrt{n}\int_{-\infty}^{\infty}(\hat{F}_n(x) - F(x)J(f(x))dx \qquad (8)$$

$$= -\frac{1}{\sqrt{n}}\sum_{1}^{n}\int_{-\infty}^{\infty}(H_i(x) - F(x))J(F(x))dx$$

where

$$H_i(x) = \begin{cases} 0, & \text{for } x \leq x_i \\ 1, & \text{for } x > x_i \end{cases}$$

By the Central Limit Theorem, Z is asymptotically normally distributed with mean 0 and variance σ^2

If J puts no weight on the extreme observations, then the condition (4) may be dropped.

Let $\xi = F^{-1}(\alpha)$ and $n = F^{-1}(\beta)$, $0 < \alpha < \beta$ 1.

Theorem 2. Let J be bounded and continuous a.e. F^{-1} on $[0,1]$, such that

$J(u) = 0$ for $0 < \mu < \alpha$ and $\beta < \mu < 1$.

If ξ and η are uniquely defined and $\sigma^2 > 0$, then

$$\sqrt{n}(L_n'' - \mu) \xrightarrow{L} N(0, \sigma^2) \text{ as } n \to \infty \quad .$$

Proof: We may substitute ξ and η for the lower and upper limits, respectively, of the integrals in the definitions of μ and σ^2. With these substitutions the asymptotic convergence of (7) holds without condition (4).

Theorem 2 is useful in establishing the asymptotic distribution of an L-estimate, such as the trimmed mean.

If J is Lipschitz of order α, where $0 < \alpha < 1$, then we may substitute for (4) the weaker condition that

$$\int_{-\infty}^{\infty}F(x)\ (1-F(x))^{(\frac{1+\alpha}{2})}\ dx < \infty$$

Theorem 3. Let J be bounded and continuous a.e. F^{-1} on $[0,1]$. If J is Lipschitz of order α $(0 < \alpha < 1)$, (9) holds and $\sigma^2 > 0$, then:

$$\sqrt{n}(L_n'' - \mu) \xrightarrow{L} N(0, \sigma 2) \text{ as } n \to \infty$$

Proof: From (10) it follows that

$$\int_{-\infty}^{\infty} (F(x))\ (1-F(x))\ d\ x < \infty.$$

Hence, $E|X_i| < \infty$. From (5) and the Lipschitz property of J we have

$$(L_n'' - \mu) = -\sqrt{n}\int_{-\infty}^{\infty}\Big[F_n(x) - F(x))\cdot (J(F(x)) + 0\,(|\hat{F}_n(x) - F(x)|^{\alpha}\Big]dx \qquad (10)$$

Now,

$$E(\sqrt{n}\int_{-\infty}^{\infty}|F_n(x) - F(x)|^{1+\alpha}dx) \qquad (11)$$

$$\leq \sqrt{n}\int_{-\infty}^{\infty}\Big[E(\hat{F}_n(x) - F(x))^2\Big]^{\frac{1+\alpha}{2}}dx$$

$$= \sqrt{n}^{-\alpha/2}\int_{-\infty}^{\infty}(F(x))\ (1-F(x))^{\frac{1+\alpha}{2}}\ dx$$

$$\to 0 \text{ as } n \to \infty$$

Therefore, $\sqrt{n}(L_n'' - \mu)$ is asymptotically distributed as,

$$\sqrt{n}\ (\int_{-\infty}^{\infty}(F(x) - \hat{F}_n(x))J(F(x))dx$$

2.2. Variable Distributions

Let $X_1, X_2, \ldots, X_n$ be independent random variables with c.d.f. $F_1, F_2, \ldots, F_n$ respectively. Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the ordered sample values. Let

$$\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} F_i(x)$$

We assume that $\tilde{F}_n(x)$ tends to a limiting distribution $\tilde{F}(x)$, say, for each x as $n \to \infty$. Since both $\tilde{F}_n(x)$ and $\tilde{F}(x)$ are non-decreasing in x, the convergence is uniform in x. Let

$$\tilde{H}_n(x) = \frac{1}{n}\sum_1^n H_i(x)$$

be the sample cdf where $H_i(x)$ is defined in (8). Clearly, $E(\tilde{H}_n(x)) = \tilde{F}_n(x)$. It follows from the strong law of large numbers that as $n \to \infty$,

$$\tilde{H}_n(x) - \tilde{F}_n(x) \xrightarrow{wp1} 0$$

and so

$$\tilde{H}_n(X) - \tilde{F}_n(x) \xrightarrow{wp1} 0 \qquad \text{for each } x.$$

Since $\tilde{H}_n(x)$, $\tilde{F}_n(x)$ and $\tilde{F}(x)$ are non-decreasing in x, the convergence is uniform in x. Hence,

$$\sup_{-\infty < X < \infty} \left|H_n(\hat{x}) - \tilde{F}_n(x)\right| \xrightarrow{wp1} 0 \qquad (11)$$

and

$$\sup_{-\infty < X < \infty} \left|\tilde{H}_n(x) - \tilde{F}(x)\right| \xrightarrow{wp1} 0 \quad \text{as } n \to \infty .$$

We now need the following assumptions:

Assumption 1. There exists a positive number N, such that, for sufficiently large n,

$$\sqrt{n}\int_{-\infty}^{\infty} |\tilde{F}_n(x) - F(x)|\, dx \le N .$$

Assumption 2. There exists a function Q and positive numbers a and b $(0 < b < 1)$ such that, $0 \le Q(x) \le 1$, $Q^b(x)$ is integrable and for sufficiently large n, $\tilde{F}_n(x) \le Q^2(x)$ for $x \le -a$ and $1 - \tilde{F}_n(x) \le Q^2(x)$ for $x \ge a$.

From assumption 2, it follows that there exists a positive number M, such that

$$\int_{-\infty}^{\infty} ((\tilde{F}_n(x))\ (1 - \tilde{F}_n(x))^{\frac{1}{2}}\ dx \le M$$

for sufficiently large n and that

$$\int_{-\infty}^{\infty} (\tilde{F}(x)\ (1-\tilde{F}(x))^{\frac{1}{2}} dx < \infty .$$

From assumptions (1) and (2) we now have

$$E(\sqrt{n}\int_{-\infty}^{\infty} |\tilde{H}_n(x) - \tilde{F}(x)|dx \le \sqrt{n}$$

$$\int_{-\infty}^{\infty}(E(\tilde{H}_n(x) - \tilde{F}(x))^2)^{\frac{1}{2}} dx$$

$$\le \sqrt{n}\int_{-\infty}^{\infty} \left[\frac{1}{n^2}\sum_1^n F_i(x)\ (1-F_i(x)) + (\tilde{F}_n(x) - \tilde{F}(x))^2\right]^{\frac{1}{2}} dx$$

$$\le \int_{-\infty}^{\infty} \left[\frac{1}{n^2}\sum_1^n F_i(x)\ (1-F_i(x)\right]^{\frac{1}{2}} dx +$$

$$\sqrt{n}\int_{-\infty}^{\infty} |\tilde{F}_n(x) - \tilde{F}(x)|dx$$

$$\le \int_{-\infty}^{\infty} (\tilde{F}_n(x)\ (1 - \tilde{F}(x))^{\frac{1}{2}} dx +$$

$$\sqrt{n}\int_{-\infty}^{\infty} |\tilde{F}_n(x) - \tilde{F}(x)|dx$$

$\le M + N$ for sufficiently large n.

Assumption 3. As $n \to \infty$,

$$\sqrt{n}\int_{-\infty}^{\infty} (\tilde{F}(x) - \tilde{F}_n(x))\ J(\tilde{F}(x))dx \to c$$

for some constant c, such that $-\infty < c < \infty$.

Let

$$\tilde{\mu} = \int_{-\infty}^{\infty} x\ J(\tilde{\tilde{F}}(x))d\tilde{F}(x)$$

$$= \int_{-\infty}^{\infty} \psi\ (\tilde{F}(x))dx + c\int_{-\infty}^{\infty} x\ d\tilde{F}(x),$$

and

$$L_n'' = \sum_1^n \left(\int_{\frac{i=1}{n}}^{\frac{1}{n}} J(u)du\right)x_i$$

$$= \int_{-\infty}^{\infty} \psi\ (\tilde{H}_n(x))dx + c\int_{-\infty}^{\infty} x\ d\tilde{H}_n(x), \text{ and}$$

$$\tilde{Z} = -\frac{1}{\sqrt{n}}\sum_1^n \int_{-\infty}^{\infty} (H_i(x) - \tilde{F}(x))J(\tilde{F}(x))dx$$

$$= -\frac{1}{\sqrt{n}}\sum_1^n \int_{-\infty}^{\infty} (H_i(x) - \tilde{F}_i(x))J(\tilde{F}(x))dx +$$

$$\sqrt{n}\int_{-\infty}^{\infty} (\tilde{F}(x) - \tilde{F}_n(x))J(\tilde{F}(x))dx$$

The variance of the ith term in the above summation is equal to

$$\sigma_i^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(\tilde{F}(x))J(\tilde{F}(y)) (\min(F_i(x), F_i(y)) - F_i(x)F_i(y))dxdy$$

Corresponding to (6), we have:

$$\sqrt{n}(L_n'' - \mu) = - \sqrt{n} \int_{-\infty}^{\infty} (\tilde{H}_n(x) - \tilde{F}(x)) (J(\tilde{F}(x)) - \tilde{H}_n(x) - \tilde{F}(x))dx \qquad (12)$$

From (11) and assumption 2, it follows that $\sqrt{n}(L_n'' - \mu)$ is asymptotically equivalent to $\tilde{Z}$, which in turn is asymptotically distributed as

$$- \frac{1}{\sqrt{n}} \sum_1^n \int_{-\infty}^{\infty} (H_i(x) - \tilde{F}(x))J(\tilde{F}(x))dx + c$$

in view of assumption 3. Let

$$R = \int_{|x|>a} Q^b(x)\,|J(F(x))|dx \text{ and}$$

$$Y_i = \int_{-\infty}^{\infty} (H_i(x) - F_i(x))J(\tilde{F}(x))dx$$

$$= \int_{|x|\le a} (H_i(x) - F_i(x))J(\tilde{F}(x))dx + \int_{|x|\le a} (H_i(x)-F_i(x))J(\tilde{F}(x))dx .$$

As $Q^b(x)$ is integrable and J is bounded, we have $R < \infty$. Now,

$$|Y_i|^{\frac{2}{b}} \le 2^{\frac{2}{b}-1}\Big[\Big(\int_{|x|>a} |H_i(x) - F_i(x)|\,|J(\tilde{F}(x))|\, dx\Big)^{\frac{2}{b}} + \Big(\int_{|x|\le a} |H_i(x) - F_i(x)|\,|H(F(x))|\, dx\Big)^{\frac{2}{b}}\Big]$$

$$\le 2^{\frac{2}{b}-1} |R^{\frac{2}{b}-1} \int_{|x|<a} \frac{|H_i(x) - F_i(x)|^{\frac{2}{b}}}{Q^2(x)} Q^b(x)\,|J(\tilde{F}(x)dx + \int_{|x|\le a} |J(\tilde{F}(x))|dx)^{\frac{2}{b}} |$$

$$\le (2R)^{\frac{2}{b}-1} \int_{|x|\ge a} \frac{(H_i(x)-F_i(x))^2}{Q^2(x)} Q^b(x)\,|J(\tilde{F}(x)|dx + S$$

Where,

$$S = (2R)^{\frac{2}{b}-1} \Big(\int_{|x|\le a} |J(F(x)|dx\Big)^{\frac{2}{b}}.$$

Hence,

$$E|Y_i|^{\frac{2}{b}} \le (2R)^{\frac{2}{b}-1} \int_{|x|>a} \frac{F_i(x)(1-F_i(x))}{Q^2(x)} Q^b(x)\,|J(F(x)|dx + S$$

$$\le (2R)^{\frac{2}{b}-1} \int_{|x|>a} Q^b(x)\,|J(\tilde{F}(x))|dx + S$$

for sufficiently large n, by assumption 2,

$$\le 2^{\frac{2}{b}-1} R^{\frac{2}{b}} + S$$

From (13), it is seen that the Liapounov's condition for the Central Limit Theorem is satisfied for the sum (12) if as $n \to \infty$.

$$\frac{1}{n} \sum_1^n \sigma_i^2 \to \tilde{\sigma}^2 > 0 \qquad (14)$$

for some positive number $\tilde{\sigma}$. Therefore, $\sqrt{n}\,(L_n'' - \tilde{\mu})$ is asymptotically normally distributed. Thus, we have:

<u>Theorem 4</u>. Let J be bounded and continuous a.e. F^{-1} on $[0,1]$. If $\tilde{F}_n(x) \to \tilde{F}(x)$ for each x and assumptions 1, 2 and 3 are satisfied and (14) holds, then

$$\sqrt{n}\,(L_n - \tilde{\mu}) \xrightarrow{L} N(c, \tilde{\sigma}^2)$$

as $n \to \infty$.

The analogue of Theorem 2 for variable distributions is given below:

<u>Theorem 5</u>. Let J be bounded and continuous a.e. $\tilde{F}^{-1}$ on (α, β) such that

$J(u) = 0$, for $0 < u < \alpha$ and $\beta < u < 1$.

If $\tilde{F}_n(x) - \tilde{F}(x)$ for each x, the αth and βth quantiles of $\tilde{F}(x)$ are uniquely defined, assumptions 1, 2 and 3 are satisfied and (14) holds, then $\sqrt{n}(L_n'' - \tilde{\mu}) \to N(c, \tilde{\sigma}^2)$ as $n \to \infty$.

3. Application

Consider the linear model:

$$Y_i = a + bx_i + t_i, \; i = 1, \ldots n$$

Assume that in this model, t_i are iid F(.) where F(.) is assumed symmetric and absolutely continuous. Padua (1986), gave the following robust estimate of b:

$$b^* = \underset{i,j \in R}{\text{median}} \frac{Y_j - Y_i}{x_j - x_i} \qquad (14)$$

where R is the set of subscripts chosen from (1, 2, ..., n) taken two at a time and having no component in common. In other words, pair off the data points and then for each pair compute the slopes $\hat{b}_1, \ldots, \hat{b}_m$ say. Compute for the median of these slopes.

One possible modification of (14) is to consider estimators of the form:

$$L_m = \sum_1^m c_i \hat{b}_{ci} \qquad (15)$$

where c_i are constants and $\hat{b}_{(1)} \leq \hat{b}_{(2)} \leq \hat{b}_{(3)} \leq \cdots \leq b_{(m)}$. The slopes $\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_m$ provide m independent estimates of b and so the asymptotic normality of the estimator (15) follows from our previous discussions.

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